Math 249, Monday April 20

Hall-Littlewood Polynomials

Recall

$$x_1^{\lambda_1} x_2^{\lambda_2} \cdots$$

$$s_\lambda(x_1,\dots,x_n) = \sum_{w \in S_n} w\left(\frac{x^\lambda}{\prod_{i<j} (1 - x_j/x_i)} \right) = \frac{a_{\lambda+\rho}}{a_\rho}$$

Define

$$H_\lambda(x_1,\dots,x_n) = \sum_{w \in S_n} w\left(\frac{x^\lambda}{\prod_{i<j} \left((1-x_j/x_i)(1-q x_i/x_j)\right)} \right)_{pol}$$

$(-)_{pol}$ means:

Expand as power series in q : sum of terms $q^m s_\mu(x)$: $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \in \mathbb{Z}^n$
Then discard any terms with some $\mu_i < 0$.

I.e. take 'raising' series $\dfrac{x^\lambda}{\prod_{i<j}(1-q x_i/x_j)}$

and map $x^\mu \mapsto s_\mu(x)$ (with usual rule if μ is arbitrary), then kill non-pol terms.

If some $\mu_k + \cdots + \mu_n < 0$, then $(\mu+\rho)_+ - \rho \geq \mu$ has a negative tail,

$\Rightarrow s_\mu(x)$ is non-polynomial.

$\rho = (n-1, n-2, \dots, 0)$
$[\, (\mu+\rho)_+ \geq \mu+\rho \,]$

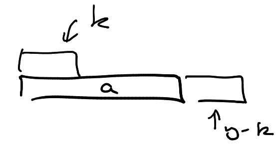
Ex. $H_{(k)}(x;q) = s_{(k)} = h_k$

$$\frac{x_1^k}{\prod_{i<j}(1-q x_i/x_j)} = x_1^k + (\text{terms with negative powers of } x_2,\dots,x_n)$$

Ex. $H_{(21)}(x;q) = S_{(21)} + q S_{(3)}$ $\qquad \dfrac{x_1^2 x_2}{\prod_{i<j}(1 - q x_i/x_j)} = x_1^2 x_2 + q x_1^3 +$ terms with negative tails

$H_{(ab)}(x;q) =$ $\qquad 1 + q x_1/x_2 + q^2 x_1^2/x_2^2 + \cdots$

$a \geq b$ $\qquad S_{(ab)} + q S_{(a+1,b-1)} + q^2 S_{(a+2,b-2)} + \cdots + q^b S_{(a+b)}$



$S(a+b-k, k)$

$0 \leq k \leq b$

Note $H_{(ab)}(x;1) = h_a h_b = h_{(ab)}$.

Def. $K_{\lambda\mu}(q) = \langle s_\lambda, H_\mu(x;q) \rangle$

Formula: Get s_λ term in $H_\mu(x;q)$ for any $x^{\omega(\lambda+\rho)-\rho}$ term in $\dfrac{x^\mu}{\prod_{i<j}(1 - q x_i/x_j)}$

x^θ : $\theta + \rho \to (\theta+\rho)_+ = \omega^{-1}(\theta+\rho)$

$\downarrow -\rho$

$(\theta+\rho)_+ - \rho$ $\qquad \omega^{-1}(\theta+\rho) - \rho = \lambda$ $\qquad \omega^{-1}(\theta+\rho) = \lambda + \rho$

with coefficient $(-1)^\omega = (-1)^{\ell(\omega)} = \varepsilon(\omega)$ $\qquad$ or $\theta + \rho = \omega(\lambda+\rho)$ $\qquad \theta = \omega(\lambda+\rho) - \rho$

Define $p_q(\theta) = \langle x^\theta \rangle \dfrac{1}{\prod_{i<j}(1 - q x_i/x_j)}$ $\qquad e_i - e_j \quad i<j$

x_1/x_3 $\qquad \dfrac{1}{1 - q x_1/x_3} = 1 + q x_1/x_3 + \cdots$

Ex. $p_q(1\,0\,-1) = q + q^2$ $\qquad e_1 - e_3 \qquad (e_1 - e_2) + (e_2 - e_3) \qquad x_1/x_3 = x_1/x_2 \cdot x_2/x_3$

$(1 + q x_1/x_2 + \cdots)(1 + q x_2/x_3 + \cdots)$

$(1 + q x_1/x_3 + \cdots)$

$K_{\lambda\mu}(q) = \sum_\omega (-1)^\omega p_q(\omega(\lambda+\rho) - \mu - \rho)$ $\qquad \langle x^{\omega(\lambda+\rho)-\rho} \rangle \dfrac{x^\mu}{\prod \cdots}$

$\uparrow \omega \in S_{\ell(\mu)}$ is enough: $\qquad \langle x^{\omega(\lambda+\rho)-\rho-\mu-\rho} \rangle \dfrac{1}{\prod \cdots} = p_q(\omega(\lambda+\rho) - \mu - \rho)$

$K_{\lambda\mu}(q) = 0$ if $\ell(\lambda) > \ell(\mu)$

Ex. Compute $H_{(111)}(x;q)$ $\qquad K_{(3),(111)}(q) = \sum_\omega (-1)^\omega p_q(\omega(510) - 321)$ $\qquad \lambda = (300) \quad \rho = (210)$

$S_{(3)}, S_{(21)}, S_{(111)}$ $\qquad = q^2 + q^3 - q^2 = q^3$ $\qquad \mu + \rho = 321$ $\quad$ 1-10, 01-1, 10-1, 1-10

$\omega(510) = 510$

$p_q(2\,-1\,-1) = q^2 + q^3$

$\omega(510) = 501 \qquad p_q(2,-2,0) = q^2$

$501 - 321 = 2,-2,0$

$H_{(111)}(x;q)$

$= s_{(111)} + (q+q^2)s_{(21)} + q^3 s_{(3)}.$

$H_{(111)}(x;1) = s_{(111)} + 2s_{(21)} + s_{(3)} = h_1^3$

3 12
2 13

$K_{(21),(111)}(q) = q+q^2$

$K_{(111),(111)}(q) = 1$

$\sum (-1)^w P_q(w(420) - 321)$

$w(420) = 420$, $420 - 321 = 10{-}1$, $P_q(10{-}1) = q+q^2$

$\sum (-1) P_q(w(321) - 321)$, $w = 1$, $P_q(000) = q^0 = 1$

$\cancel{w(420) = 402,\ 402 - 321 = 1{-}21}$

$K_{\lambda\mu}(1) = K_{\lambda\mu} \underset{\mu=(1^n)}{=} SYT(\lambda)$

$K_{\lambda,\mu}(q) = \frac{1}{1-x_j/x_i}$

$\langle s_\lambda, h_1^{i\le j}\rangle$

Recall $K_{\lambda\mu} \underset{def}{=} \langle s_\lambda, h_\mu\rangle = |SSYT(\lambda,\mu)| = \langle x^\mu\rangle s_\lambda \overset{?}{=} K_{\lambda\mu}(1)$

$$s_\lambda = \frac{a_{\lambda+\rho}}{a_\rho} = \frac{\sum_w' (-1)^w w(x^{\lambda+\rho})}{x^\rho \prod_{i<j}(1-x_j/x_i)}$$

$$K_{\lambda\mu} = \langle x^\mu\rangle s_\lambda = \sum (-1)^w \langle x^\mu\rangle \frac{x^{-\rho} w(x^{\lambda+\rho})}{\prod_{i<j}(1-x_j/x_i)}$$

$$\sum (-1)^w \langle x^{\mu+\rho-w(\lambda+\rho)}\rangle \frac{1}{\prod_{i<j}(1-x_j/x_i)}.$$

$$= \sum (-1)^w \langle x^{w(\lambda+\rho)-\mu-\rho}\rangle \underbrace{\frac{1}{\prod_{i<j}(1-x_i/x_j)}}_{P_1(w(\lambda+\rho)-\mu-\rho)}$$

$$K_{\lambda\mu} = \sum (-1)^w P_{q=1}(w(\lambda+\rho)-\mu-\rho)$$

$$K_{\lambda\mu}(q) = \sum (-1) P_q(w(\lambda+\rho)-\mu-\rho)$$

$\Rightarrow K_{\lambda\mu}(1) = K_{\lambda\mu}$

i.e. $H_\mu(x;1) = h_\mu$

also $H_\mu(x;0) = s_\mu$

$K_{\lambda\mu}(q) \in \mathbb{N}[q]$ (non-negative coefficients)

is q-analog of $|SSYT(\lambda,\mu)|$

$\overset{?}{=} \sum_{T\in SSYT(\lambda,\mu)} q^{c(T)} \leftarrow$ Yes: $c(T)$ = "charge" of T (Lascoux - Schützenberger '70's)

↑ 2 partition

Proof completed by Lynn Butler '80's (complicated, using Morris recurrences)
Easier these days. (via Macdonald polynomials)

What's charge?

3	5		
2	3	4	
1	1	1	2

$\lambda = (4,3,2)$ $\mu = (3,2,2,1,1)$

5
2 3 4
1 1 1 2 3

$\overrightarrow{\text{row reading word } W}$ 3 5 2 3 4 1 1 1 2 $c(T) \underset{\text{def}}{=} c(W)$

Scan R to L $\longleftarrow$

circularly, looking

for 1, 2, 3, 4, 5

$3_0 5_1 2_0 3\, 4_1\, 1\, 1\, 1_0 2$ ← first scan

$3_0 5_1 2_0 3_1 4_1\, 1\, 1_0 1_0 2_1$ ← 2nd scan

$3_0 5_1 2_0 3_1 4_1 1_0 1_0 1_0 2_1$ ← 3rd scan

$c(W) = \Sigma$ of labels $= 4$

$K_{(n),(1^n)}$ = $\binom{n}{2}$

$\boxed{1\ 2\ 3\ 4\ 5\ 6\ 7}$

$1_0\, 2_1\, 3_2\, 4_3\, 5_4\, 6_5\, 7_6$

$c(T) = \binom{n}{2}$

Jeu-de-Taquin invariant Knuth relations $acb \sim cab$ or $bac \sim bca$

reverse is

$x\,W$
$\downarrow$
$W\,x$ $x \neq 1$

$c(Wx) = c(xW)+1$

$cc(Wx) = cc(xW) - 1$

$a\,b\,c$
$\downarrow$
ba

$a_y\ a{+}1_x\ a_x$
$a{+}1_x\ a_y\ a_x$

$a\,ba$ or $ba\,b$

$a<b$

$a{+}1_y\ a{+}1_x\ a_x$
$a{+}1_x\ a_x\ a{+}1_y$

$a\ \ a{+}1\ \ a_x$
$a\ \ a{+}1$

$a\ c$ $c\ a$

$a<b<c$

2 4

1 1 1 2 2 3 3 4 5